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The space of interactions in neural networks with hierarchical cluster organization

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Abstract. We study the storing capacity of a neural network with a synapsis organized in two clusters, by analysing the maximal volume in interaction space. The cluster organization is introduced through a modified spherical condition on the interactions and also through the requirement of storing together a family of patterns formed by an 'ancestor' and one 'descendant' that differ on the relative sign of the cluster configurations. The critical capacity α^c is compared with Gardner's result for a uniform system, α_G, with the result that when good retrieval of only one member of the 'family' is required the ratio α^c/α_G coincides with the value obtained previously by the signal-to-noise method. When we analyse the volume corresponding to joint retrieval of 'ancestor' and 'descendant' we obtain $\alpha^c/\alpha_G = \frac{1}{2}$ regardless of the cluster modulation.

1. The model and results

In a previous publication [1] we presented a model for neural networks where the N sites of the network are organized in ℓ clusters with hierarchical interactions. The energy function is given by

$$\mathcal{H} = -\frac{1}{2}\sum_{a,b}^{\ell}\sum_{i}^{(a)}\sum_{j}^{(b)} J_{ij}^{ab}\sigma_i\sigma_j \tag{1}$$

where the i, j indicate network sites and $a, b = 1, \ldots, \ell$ are cluster indices. The synaptic junctions were defined as

$$J_{ij}^{ab} = A_{ab}(\ell) J_{ij}^{\mathrm{Hebb}} \qquad i \in a \quad j \in b \tag{2}$$

in terms of Hebb's learning rule

$$J_{ij}^{\mathrm{Hebb}} = \frac{1}{\sqrt{N}}\sum_{\mu}^{p}\xi_i^{\mu}\xi_j^{\mu} \tag{3}$$

and the elements $A_{ab}(\ell)$ of an $\ell \times \ell$ matrix $\underline{\underline{A}}$ that has ultrametric structure, given by the recursion relation

$$\underline{\underline{A}}(\ell) = \begin{bmatrix} \epsilon\underline{\underline{A}}(\ell/2) + \underline{\underline{U}}(\ell/2) & \underline{\underline{U}}(\ell/2) \\ \underline{\underline{U}}(\ell/2) & \epsilon\underline{\underline{A}}(\ell/2) + \underline{\underline{U}}(\ell/2) \end{bmatrix} \tag{4}$$

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where $\underline{\underline{U}}(d)$ is a d-dimensional matrix with all elements equal to unity and $\underline{\underline{A}}(1) = 1$. Hopfield's model is recovered for $\epsilon = 0, A_{ab}(\ell) \equiv 1$, while for large values of ϵ the interactions are concentrated within one cluster [3]. As a consequence of the spatial modulation we found that for every stored pattern $\{\xi_i^\mu\}$ the network also retrieves a family [2] of 'descendants'

$$\eta_i^{\mu\gamma} \equiv v_a^\gamma \xi_i^\mu \qquad i \in a \tag{5}$$

where the $v_a^\gamma = \pm 1$ are the components of an eigenvector of matrix $\underline{\underline{A}}$ with eigenvalue λ_γ, $\gamma = 1, \ldots, \ell$. The largest eigenvalue λ_1 corresponds to the 'ancestor' or stored pattern, with $v_a^1 \equiv 1$. Hopfield's model is recovered from (1) when A_{ab} = constant and gives $\lambda_1 > 0$, $\lambda_2 = \lambda_3 = \cdots = 0$.

The saturation properties of the network for an extensive number $p = \alpha N$ of stored patterns were analysed in [3], with the following results.

(i) The signal-to-noise analysis gives for the ratio of critical capacities

$$\alpha_\gamma^{\rm SN} / \alpha_{\rm H}^{\rm SN} = \lambda_\gamma^2 \Big/ \left(\sum_{\delta=1}^{\ell} \lambda_\delta^2 \right) \tag{6}$$

where $\alpha_{\rm H}^{\rm SN}$ corresponds to Hopfield's model and $\alpha_\gamma^{\rm SN}$ is the critical capacity for the storage of the γ-descendant in (5).

(ii) A mean-field theory calculation [3] gives the following bound:

$$\frac{\alpha_1^{\rm SN}}{\alpha_{\rm H}^{\rm SN}} \leqslant \frac{\alpha_1^{\rm c}}{\alpha_{\rm H}^{\rm c}} \leqslant 1$$

where $\alpha_1^{\rm c}$ is the critical capacity for the stored pattern and $\alpha_{\rm H}^{\rm c}$ the corresponding quantity in Hopfield's model.

In this paper we present a complementary analysis of the problem by using Gardner's method [4]. We ask the question of what is the volume in interaction space such that the patterns in (5) are attractors of the dynamical equations, equivalent to satisfying the relations

$$\eta_i^{\mu\gamma} h_i^{\mu\gamma} > \kappa_\gamma \tag{7}$$

where the local field is given by

$$h_i^{\mu\gamma} = \frac{1}{\sqrt{N}} \sum_j J_{ij} \eta_j^{\mu\gamma} \tag{8}$$

and the parameter $\kappa_\gamma \geqslant 0$ is related to the size of the basin of attraction for each descendant. The interactions are in general non-symmetric, $J_{ij} \neq J_{ij}$, and the spatial modulation is introduced through a modified spherical condition, similar to that satisfied by the synapsis in (2):

$$\sum_{j \in b} J_{ij}^2 = A_{ab}^2(\ell) \frac{N}{\ell} \qquad \text{if} \quad i \in a\,. \tag{9}$$

The study of the problem for arbitrary values of ℓ is very complex, hence we restrict this analysis to two clusters, $\ell = 2$. The matrix $\underline{\underline{A}}$ in (4) becomes 2×2 and we indicate by A_1 (A_2) the diagonal (off-diagonal) elements, $A_1 > A_2$. The patterns together with their only descendant in (5) are

$$\eta_i^{\mu 1} = \xi_i^\mu \qquad \eta_i^{\mu 2} = \begin{cases} \xi_i^\mu & 1 \leqslant i < N/2 \\ -\xi_i^\mu & N/2 \leqslant i \leqslant N\,. \end{cases} \tag{10}$$

Following closely Gardner's work [4] we introduce the volume in interaction space

$$\Omega(\alpha, \kappa_1, \kappa_2) = \int \prod_{j \neq i} \mathrm{d}J_{ij} \prod_\mu \theta\left(\eta_i^{\mu 1} h_i^{\mu 1} - \kappa_1\right) \theta\left(\eta_i^{\mu 2} h_i^{\mu 2} - \kappa_2\right) \times \delta\left(\sum_{j \in 1} J_{ij}^2 - A_1^2 \frac{N}{2}\right) \delta\left(\sum_{j \in 2} J_{ij}^2 - A_2^2 \frac{N}{2}\right) \tag{11}$$

where we have exploited the independence of $J_{ij} \neq J_{ji}$ and without loss of generality we assume that $i \leqslant N/2$ belongs to cluster 1, since for $i > N/2$ we will obtain the same equations by symmetry. We do not write the normalization constant that is irrelevant in the following discussion. The quantity we are interested in is the configurational average of $\ln \Omega$ over the random patterns, that we calculate by using the replica method. Then

$$\langle \ln \Omega \rangle = \lim_{n \to 0} \frac{\Omega_n - 1}{n} = GN \tag{12}$$

where calling $\rho = 1, \ldots, n$ the replica index, we have

$$\Omega_n \equiv \langle \Omega^n \rangle = \int \prod_\rho \left[\prod_j \mathrm{d}\, J_{ij}^\rho \prod_{b=1}^{2} \delta\left(\sum_{j \in b} (J_{ij}^\rho)^2 - A_b^2 \frac{N}{2} \right) \right] \times \left\langle \prod_{\mu, \rho} \prod_{\gamma=1}^{2} \theta\left(\eta_i^{\mu\gamma} h_{i\rho}^\mu - \kappa_\gamma \right) \right\rangle. \tag{13}$$

We indicate by $h_{i\rho}^{\mu\gamma}$ in (13) the local field in (8) in terms of J_{ij}^ρ, and $\eta_i^{\mu\gamma}$ was defined in (10). The method of calculation was described by Gardner [4] and we refer the reader to this paper for details. We introduce the order parameters

$$q_b^{\rho\rho'} = \frac{2}{N} \sum_{j \in b} J_{ij}^\rho J_{ij}^{\rho'} \qquad \rho \neq \rho' \tag{14}$$

when $i \leqslant N/2$ belongs to cluster 1 and $b = 1$ or 2, and by using the integral representation for the step function we obtain in (13)

$$\Omega_n = \int \prod_{j, \rho} \mathrm{d}\, J_{ij}^\rho \prod_b \int \prod_{\rho \neq \rho'} \mathrm{d}q_b^{\rho\rho'} \delta\left(\sum_{j \in b} J_{ij}^\rho J_{ij}^{\rho'} - \frac{N}{2} q_b^{\rho\rho'} \right) \prod_\rho \times \delta\left(\sum_{j \in b} (J_{ij}^\rho)^2 - A_b^2 \frac{N}{2} \right) \Lambda^{\alpha N} \tag{15}$$

where

$$\Lambda = \int_{(\kappa_1+\kappa_2)/\sqrt{2}}^{\infty} \prod_{\rho} \frac{d\lambda_1^{\rho}}{2\pi} \int_{\sqrt{2}\kappa_1-\lambda_1}^{\lambda_1-\sqrt{2}\kappa_2} \prod_{\rho} \frac{d\lambda_2^{\rho}}{2\pi} \int_{-\infty}^{\infty} \prod_{\rho} dx_1^{\rho} dx_2^{\rho}$$

$$\times \exp\left\{\sum_b \left[\sum_{\rho}\left(\mathrm{i}x_b^{\rho}\lambda_b^{\rho} - \frac{1}{2}A_b^2 x_b^{\rho^2}\right) - \frac{1}{2}\sum_{\rho\neq\rho'} q_b^{\rho\rho'} x_b^{\rho} x_b^{\rho'}\right]\right\}. \tag{16}$$

The next step is to use the integral representation of the δ-functions in (15), and to reach an integral expression for Ω_n that can be solved by the saddle-point method [4]. Assuming from the start a replica symmetric solution

$$q_b^{\rho\rho'} = q_b \qquad \rho \neq \rho' \tag{17}$$

the Gaussian integrals over J_{ij}^{ρ} and x_b^{ρ} are easily performed with the result at the saddle point, when $n \to 0$:

$$\ln \Omega_n = +nN\left[\sum_b \left(\frac{1}{4}\frac{F_b}{E_b - F_b} + \ln(E_b - F_b) - \frac{\mathrm{i}}{2}F_b q_b + \frac{\mathrm{i}}{2}E_b A_b^2\right)\right.$$

$$\left. + \alpha \int Dz_1\, Dz_2 \ln W\right]. \tag{18}$$

where we used the notation

$$Dz = \frac{dz}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \tag{19}$$

and

$$W = \int_{(\kappa_1+\kappa_2)/\sqrt{2}}^{\infty} \frac{d\lambda_1}{2\pi} \int_{\sqrt{2}\kappa_1-\lambda_1}^{\lambda_1-\sqrt{2}\kappa_2} \frac{d\lambda_2}{2\pi} \prod_b \frac{1}{\sqrt{\epsilon_b}} \exp\left[-\left(\lambda_b + \sqrt{q_b}z_b\right)^2/2\epsilon_b\right] \tag{20}$$

with

$$\epsilon_b = A_b^2 - q_b. \tag{21}$$

After solving the saddle-point equations for E_b and F_b we can write in (12) from (18):

$$G = \frac{1}{4}\sum_b \left(\frac{A_b^2}{A_b^2 - q_b} + \ln\left(A_b^2 - q_b\right)\right) + \alpha \int Dz_1\, Dz_2 \ln W. \tag{22}$$

1.1. Limit $\kappa_2 \to -\infty$

Before proceeding with the general saddle-point equations for q_b, we discuss the simpler case of storing only one pattern per family in (11).

When $\kappa_2 \to -\infty$ we are left with the volume of space modulated interactions that satisfy the block equations (9), but that allow only $\eta^{\mu 1}$ in (10) to be an attractor of the dynamics. In this limit we get from (20)

$$W(\kappa_2 = -\infty) = H\left[\frac{\sqrt{2}\kappa_1 + \sqrt{q_1}z_1 + \sqrt{q_2}z_2}{(\epsilon_1 + \epsilon_2)^{\frac{1}{2}}}\right] \tag{23}$$

where we introduced the function

$$H(x) = \int_x^{\infty} \mathrm{D}z. \tag{24}$$

From (22) and (23) the saddle-point equations for q_1 and q_2 are

$$\frac{q_1}{\epsilon_1}\left(\frac{\epsilon_1 + \epsilon_2}{2\epsilon_1}\right) = \frac{q_2}{\epsilon_2}\left(\frac{\epsilon_1 + \epsilon_2}{2\epsilon_2}\right) = \frac{\alpha}{2\pi}\int_{-\infty}^{\infty} \mathrm{D}z_1 \mathrm{D}z_2 \mathrm{e}^{-f^2}\frac{1}{H^2(f)} \tag{25}$$

$$f = \frac{\sqrt{2}\,\kappa_1 + \sqrt{q_1}\,z_1 + \sqrt{q_2}\,z_2}{(\epsilon_1 + \epsilon_2)^{\frac{1}{2}}}. \tag{26}$$

Equation (25) indicates that for $\alpha = 0$, $q_b = 0$ and the J_{ij} that solve (7) are uncorrelated, while for increasing values of α, q_b also increases. At the critical value α_c the volume reduces to a point, there is only one choice of J_{ij} and, from (14) and the spherical condition in (9), q_b reaches its maximum value A_b^2. From (21) and the first equality in (25) we obtain that for $\alpha \to \alpha_c$, $q_b = A_b^2 - A_b\delta$, $\delta \to 0$. The integral in (25) also becomes singular and we obtain by calculating the asymptotic behaviour

$$\alpha_1^{\mathrm{c}}(\kappa_1) = \frac{1}{2}\frac{(A_1 + A_2)^2}{A_1^2 + A_2^2}\left(\int_{-\kappa'}^{\infty} \mathrm{D}z\,(\kappa' + z)^2\right)^{-1} \tag{27}$$

with $\kappa' = \kappa_1\sqrt{2}/(A_1^2 + A_2^2)^{\frac{1}{2}}$. In terms of the eigenvalues $\lambda_1 = A_1 + A_2$ and $\lambda_2 = A_1 - A_2$ of the matrix $\underline{\underline{A}}$ in (14) for $\ell = 2$, we can write (27) as

$$\alpha_1^{\mathrm{c}}(\kappa_1) = \frac{\lambda_1^2}{\lambda_1^2 + \lambda_2^2}\alpha_{\mathrm{G}}^{\mathrm{c}}(\kappa') \tag{28}$$

where we indicate by $\alpha_{\mathrm{G}}^{\mathrm{c}}(\kappa')$ Gardner's expression [4] for the storing capacity of uncorrelated patterns with bassin of attraction κ'. When $\kappa_1 = 0$ we recover from (28) the same value for the ratio among critical storing capacities, $\alpha_1^{\mathrm{c}}(0)/\alpha_{\mathrm{G}}^{\mathrm{c}}(0)$, that has been obtained from the signal-to-noise method [3] and is given in (6).

The result in (28) was obtained by letting $\kappa_2 \to -\infty$ in (11) and expresses the critical capacity for storing only the 'ancestor' in the network, corresponding to the eigenvalue $\lambda_\gamma = \lambda_1$ in (6). However, by taking the opposite limit $\kappa_1 \to -\infty$, $\kappa_2 \geqslant 0$, we still obtain the same result in (28) and *not* the equivalent to (6) with $\lambda_\gamma = \lambda_2$. This is because the local field and the spherical condition in (8) and (9) are invariant under the joint transformation $\xi_j^\mu \to -\xi_j^\mu$, $J_{ij} \to -J_{ij}$, hence the system cannot distinguish between 'ancestor' and 'descendant'. For $A_2 = A_1$ we recover the result for a non-modulated network, $\alpha_c = 2$, while for $A_2 = 0$ we are left with only half of the sites to store p patterns; then the storing capacity is reduced by a factor $\frac{1}{2}$.

1.2. General case

We obtain from (20)

$$W = \int_{\ell_1}^{\infty} \mathrm{D}y \left\{ H\left[\sqrt{\frac{\epsilon_1}{\epsilon_2}}(-y+\ell_1+\ell_2)\right] - H\left[\sqrt{\frac{\epsilon_1}{\epsilon_2}}(y-\ell_1+\ell_2)\right]\right\} \tag{29}$$

with

$$\ell_1 = \frac{1}{\sqrt{\epsilon_1}}\left(\frac{\kappa_1+\kappa_2}{\sqrt{2}} + \sqrt{q_1}z_1\right) \qquad \ell_2 = \frac{1}{\sqrt{\epsilon_1}}\left(\frac{\kappa_1-\kappa_2}{\sqrt{2}} + \sqrt{q_2}z_2\right). \tag{30}$$

The calculations are now far more involved than in the previous case. We obtain for the saddle-point equations

$$\frac{q_1}{\epsilon_1^2} = \frac{2}{\epsilon_1+\epsilon_2}\frac{\alpha}{2\pi}\int \mathrm{D}z_1\mathrm{D}z_2\frac{1}{W^2}\left(\exp\left[-\frac{1}{2}\frac{\epsilon_1}{\epsilon_1+\epsilon_2}(\ell_1+\ell_2)^2\right]H(\zeta_-)\right.$$
$$\left. + \exp\left[-\frac{1}{2}\frac{\epsilon_1}{\epsilon_1+\epsilon_2}(\ell_1-\ell_2)^2\right]H(\zeta_+)\right)^2 \tag{31}$$

$$\frac{q_2}{\epsilon_2^2} = \frac{2}{\epsilon_1+\epsilon_2}\frac{\alpha}{2\pi}\int \mathrm{D}z_1\mathrm{D}z_2\frac{1}{W^2}$$
$$\times\left(\exp\left[-\frac{1}{2}\frac{\epsilon_1}{\epsilon_1+\epsilon_2}(\ell_1+\ell_2)^2\right]H(\zeta_-)\right.$$
$$\left. - \exp\left[-\frac{1}{2}\frac{\epsilon_1}{\epsilon_1+\epsilon_2}(\ell_1-\ell_2)^2\right]H(\zeta_+)\right)^2 \tag{32}$$

where

$$\zeta_\pm = \frac{\epsilon_2\ell_1 \pm \epsilon_1\ell_2}{\sqrt{\epsilon_2(\epsilon_1+\epsilon_2)}}. \tag{33}$$

To obtain the critical storage capacity we take simultaneously the limits $\epsilon_1, \epsilon_2 \to 0$ while keeping constant the ratio $\epsilon_1/\epsilon_2 = r$, that is to be determined together with α^{c} from the solution of (31) and (32). To calculate the singular behaviour of the integrals in these equations we look for the values of z_1, z_2 that make W in (29) vanishingly small and we frequently use, for an arbitrary function $F(y)$

$$\int_x^{\infty} F(y)\,\mathrm{d}y = \theta(-x)\int_{-\infty}^{\infty} F(y)\,\mathrm{d}y - \theta(-x)\int_{|x|}^{\infty} F(-y)\,\mathrm{d}y + \theta(x)\int_x^{\infty} F(y)\,\mathrm{d}y \tag{34}$$

together with the asymptotic expansion of the last two integrals for large values of $|x|$.

The calculations are straigthforward but tediously long and we just quote the final results for the maximum storing capacity at $\kappa_1 = \kappa_2 = 0$. We obtain from (31) and (32)

$$A_1^2\left(\frac{1+r}{r}\right)^2 = \alpha\left[4A_2^2\int_0^{\infty}\mathrm{D}z_1\int_{\Delta z_1}^{\infty}\mathrm{D}z_2(z_2-\Delta z_1)^2 + A_1^2\left(\frac{1+r}{r}\right)^2\right] \tag{35}$$

$$A_2^2 r^2 = A_1^2 - 4\alpha r^2 A_2^2\int_0^{\infty}\mathrm{D}z_1\int_0^{z_1\Delta/r}\mathrm{D}z_2\left[\left(\frac{z_1\Delta}{r}\right)^2 - z_2^2\right] \tag{36}$$

where $\Delta = A_1/A_2$. For $A_2 = 0$ we recover exactly $\alpha^c = 1$, which coincides with the result obtained in (28). This is the correct limit because when $A_2 = 0$ and $\kappa_1 = \kappa_2$, the θ-functions in (11) have the same arguments and (11) coincides with (13). The integrals in (35) and (36) are standard [5] and the equations can be solved numerically for $\alpha_c(\Delta)$ and $\epsilon_1/\epsilon_2 = r(\Delta)$, with the result shown in figure 1. It is surprising that when we impose the condition of storing the two related patterns in (5), the storing capacity is not sensitive to modulation and practically sticks at its value for $A_2 = 0$.

2. Summary and conclusions

When Hebb's learning rule is replaced in Hopfield's model by the space-modulated learning algorithm in (2), we find [1–2] that the system retrieves the family of 'descendants' in (5) together with the original pattern or 'ancestor'. The network accepts the storage of an extensive number of patterns $p = \alpha N$, but the value of the critical storage capacity depends now on the descendant that is being retrieved [3]. The signal-to-noise analysis tells us that for $\alpha > \alpha_\gamma^{\mathrm{SN}}$ there is no retrieval of the γ-descendant in (5), where $\alpha_\gamma^{\mathrm{SN}}$ is given in (6) in terms of the eigenvalues of the matrix $\underline{\underline{A}}$. Hence the smaller λ_γ, the smaller the corresponding critical storage capacity $\alpha_\gamma^{\mathrm{SN}}$.

In the present paper we take a complementary approach and we analyse the space of interactions [4] of a neural network with a synapsis which is spatially modulated by the spherical condition in (9), and which has the complete 'family' of patterns in (5) as attractors of the dynamical equations. The results obtained depend strongly on the parameters κ_γ, $\gamma = 1, \ldots, \ell$, that characterize the basin of attraction for each descendant in (7).

For the sake of clarity the calculations presented here consider the simplest case of just two clusters, and the basin of attraction parameters for the 'ancestor' and 'descendant' are indicated by κ_1 and κ_2, respectively. When we take the limit $\kappa_2 \to -\infty$ in (23) we are relaxing the condition of having the 'descendant' as an attractor, and we study the volume of space-modulated interactions that allow only the 'ancestor' to have a finite basin of attraction. In this case we find that the critical storing capacity in (28) is related to Gardner's result [4] in a way analogous to the signal-to-noise expression in (6). This result is general and it can be proved [6] that for arbitrary number ℓ of clusters we obtain, when $\kappa_2, \ldots, \kappa_\ell \to -\infty$

$$\alpha_1^c(\kappa_1) = \frac{\gamma_1^2}{\sum_\gamma \lambda_\gamma^2} \alpha_G^c(\kappa')$$

and $\kappa' = \kappa_1/[(1/\ell)\sum_b A_{ab}^2]^{\frac{1}{2}}$. The result for $\alpha_1^c(0)$ and $\ell = 2$ is plotted with a broken curve in figure 1. For $A_2 = 0$ the reference site is connected to only half of the sites, then the storage capacity also decreases to one half of Gardner's value. When $A_2 = A_1$ the modulation vanishes and we recover the known result [4] $\alpha_1^c(0) = 2$.

At the opposite limit we analyse the volume of space-modulated interactions that accept 'ancestor' and 'descendant' together as attractors of the dynamical equations. This case becomes particularly involved because the ℓ patterns in (5) are not statistically independent and we present in (31) and (32) the saddle-point equations for

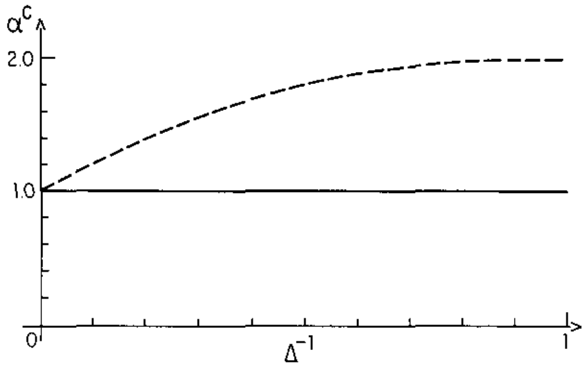

Figure 1. Critical capacity α^c for storing a 'family' of two patterns (full curve) and for storing only one pattern (broken curve) in a neural network with two clusters, as a function of the ratio $\Delta^{-1} = A_2/A_1$, where $A_b \equiv A_{1b}$ is related to the interaction strength between cluster 1 and cluster $b = 1$ or 2.

$\ell = 2$. The maximal storing capacity for $\kappa_1 = \kappa_2 = 0$ is obtained by solving numerically (35) and (36). The results shown by the full line in figure 1 indicate that $\alpha^c(0)$ sticks to one half of Gardner's value [4] independently of the ratio $\delta = A_1/A_2$.

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References

[1] Idiart M A P and Theumann A 1990 *Neural Networks and Spin-Glasses* ed W K Theumann and R Köberle (Singapore: World Scientific)
[2] Idiart M A P and Theumann A 1991 *J. Phys. A: Math. Gen.* **24** L649–58
[3] Idiart M A P and Theumann A 1992 *J. Phys. A: Math. Gen.* **25** 779–95
[4] Gardner E 1988 *J. Phys. A: Math. Gen.* **21** 257–70
[5] Gradshteyn S and Ryzhik I M 1965 *Table of Integrals, Series and Products* (New York: Academic)
[6] Idiart M A P 1992 *PhD Thesis* Universidade Federal do Rio Grande do Sul, Brazil (unpublished)